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Non-bicolourable finite configurations of rays and their deformations

Artur E Ruuge

Department of Quantum Statistics and Field Theory, Faculty of Physics,
Moscow State University, Vorobyovy Gory, 119899 Moscow, Russia
and

Department of Mathematics and Computer Science, University of Antwerp,
Middelheim Campus Building G, Middelheimlaan 1, B-2020 Antwerp, Belgium

E-mail: Artur.Ruuge@ua.ac.be

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Abstract

A new infinite family of examples of finite non-bicolourable configurations of rays in Hilbert space is described. Such configurations appear in the analysis of quantum mechanics in terms of Bell's inequalities and the Kochen–Specker theorem and illustrate that there is no measurable space in the background of the probability model of a quantum system. The mentioned examples are naturally parametrized by a positive integer divisible by 4 and by several complex-valued parameters, whose number depends on this integer. In order to compare two configurations with the same number of rays, a notion of deformation of a configuration is introduced. The constructed examples are then interpreted as obtained by way of deformations.

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1. Introduction

The non-bicolourable finite configurations of rays play a substantial role in quantum mechanics. Generally speaking, they are meant to illustrate various weird features of quantum theory—the so-called contextuality, non-locality, indeterminism, etc (depending on the metaphysical point of view accepted by a particular scientist). This makes them especially interesting, and in fact fundamental, for the quantum computing technology.

The terminology used in the present paper is close to that introduced in [1]. Let \mathcal{H} be a Hilbert space over \mathbb{C} of finite dimension d . Consider the set of all projective lines $\mathbb{P}(\mathcal{H})$, and let A be a subset of it. The elements of A are termed as *rays*. Let Γ be a finite set consisting of two formal symbols termed as *colours*, say $\Gamma := \{\text{red}, \text{blue}\}$. Denote by \perp the orthogonality relation on A induced by the inner product on \mathcal{H} . A function $v : A \rightarrow \Gamma$ is called a *bicolouring*

if the following two conditions are met: (1) for every $l, l' \in A$, if $v(l')$ and $v(l)$ are both red, then $l \not\perp l'$; (2) in every collection $l_1, l_2, \dots, l_d \in A$ of d pairwise orthogonal rays, there is an element l_r such that $v(l_r) = \text{red}$. The set A is called *bicolourable* in case it admits a bicolouring, and *non-bicolourable* otherwise.

Intuitively, given a collection of rays A , it is natural to imagine the possibility of performing a ‘homotopy process’ over it. For example, if A is non-bicolourable, is it possible to vary the positions of rays in such a way that it remains non-bicolourable? Is it possible to transform a bicolourable configuration into a non-bicolourable one? So, it is better to view the non-bicolourability as a property of the *configuration*.

The physical meaning of non-bicolourability can be illustrated as follows. Interpret \mathcal{H} as the Hilbert space associated with some quantum-mechanical system. Hence, for every $l \in A$, the corresponding orthogonal projector $\hat{\pi}_l$ represents in \mathcal{H} an observable that may acquire just two values, 0 and 1. Call this observable P_l , and denote $\mathcal{O}_A := \{P_l\}_{l \in A}$. It may seem natural from the positions of classical physics to think that it is possible to construct a non-empty set Ω and a map $\rho : \mathcal{O}_A \rightarrow \mathcal{P}(\Omega)$, such that $\rho(l')$ and $\rho(l)$ are disjoint whenever $l \perp l'$, and for every collection l_1, l_2, \dots, l_d of pairwise orthogonal rays, the sets $\rho(l_i)$, $i = 1, 2, \dots, d$, partition Ω . But in this case, one can associate with every point $\omega \in \Omega$ a bicolouring v_ω of A : $v_\omega(l) := \text{red}$, if $\rho(l) \ni \omega$, and $v_\omega(l) := \text{blue}$, otherwise. Hence, for a non-bicolourable A such a map ρ cannot exist. Therefore, the behaviour of the system with respect to the observables P_l , $l \in A$, will always look non-classical. For a detailed discussion of the physical meaning of such constructions, one may refer to [2–10]. The general motivation can be found in [11, 12].

It is interesting to mention the link between the notion of non-bicolourability and the discussion about indeterminism in quantum physics. If one speaks about ‘determinism’, one operates with the notions of *cause* and *effect*. Whatever happens always has a reason, why it happens. A *naive* conception of determinism is based on a set-theoretic understanding of ‘causes’ and ‘effects’. In other words, they are mathematically nothing more but *points* of some sets. Invoking the above notation, one is tempted to view the points $\omega \in \Omega$ as causes in the following sense. For every *single* measurement act, just one of the points $\omega \in \Omega$ becomes active; if $\rho(l) \ni \omega$, then the result of the measurement of P_l is destined to be 1; otherwise it should be 0. Since for a non-bicolourable A such a space Ω does not exist at all, the naively deterministic point of view on the system with respect to observables \mathcal{O}_A must be ruled out.

A more careful investigation of the physical meaning behind the non-bicolourable configurations should take into account the fact that the collection of all observables corresponding to the orthogonal projections onto one-dimensional subspaces is not just a *set*, but a *topological space* (the topology stems from the inner product on \mathcal{H}). It is known due to [13] that finite non-bicolourable configurations in \mathcal{H} exist whenever its dimension d satisfies $d \geq 3$. At the same time, according to [5–7], every such space admits a (countable) bicolourable configuration, which is *dense* if viewed in the mentioned topology. This implies that if we accept a thesis that no experimental setup can achieve an ideal realization of measurements (i.e. one only tries to measure a target observable, but the resulting observable being actually measured is not precisely known), then one may question the falsifiability of the existence of ‘non-contextual hidden variables’ in (non-relativistic) quantum mechanics. This has been a subject of some non-trivial discussions in recent papers (see [8–10] and references therein).

It is not difficult to give an example of a non-bicolourable configuration. If the dimension d of the space \mathcal{H} satisfies $d \geq 3$, then the whole set $\mathbb{P}(\mathcal{H})$ is non-bicolourable. This is a straightforward corollary of a classical result in functional analysis—Gleason’s theorem. More important is that there exist *finite* non-bicolourable configurations. Kochen and Specker have found [13] the first example of such a configuration for $d = 3$. Their construction is

quite sophisticated and involves 117 rays. Since then several other examples in spaces of other dimensions have been found [1, 14–19]. It is necessary to note that they all exhibit some degree of symmetry, and by that the non-bicolourability may be viewed as stemming from the properties of the corresponding group. Despite this fact, a complete classification of all finite non-bicolourable configurations is not known and so far each time an element of creativity is required to find a new example. The aim of the present paper is to describe a new *infinite* family (or, more precisely, family of families) of such configurations and to investigate the possibility of their deformations.

2. Non-orthogonality and deformations

Let A be a finite collection of rays in a complex Hilbert space \mathcal{H} of finite dimension d . In order to verify whether A is bicolourable or not, it suffices to know just the orthogonality relation between its elements, or, what is equivalent, but conceptually better, the *non-orthogonality* relation $\not\perp$. The formulae defining the rays themselves become at this stage unessential. Let us start with the description of this relation for the family of the upcoming examples.

We need some auxiliary notation first. Let V be a *finite* set, $\#V = N$. It will be necessary to assume later that the number 4 divides N , $N = 4n, n \in \mathbb{N}$, but at this moment it is not important. Now, let $p_0, p_1, p_2, p_3 \in \mathbb{Z}/2$ be four parameters. Look at all functions $\varphi : V \rightarrow \mathbb{Z}/2$ and for every $U \subset V$ denote

$$L(U) := \left\{ \varphi : V \rightarrow \mathbb{Z}/2 \mid \sum_{v \in U} \varphi(v) = p_{\#_4 U} \right\}, \tag{1}$$

where $\#_4 U \in \mathbb{Z}/4$ is the cardinality of U modulo 4. Consider the *disjoint* union of the sets $L(U)$. Denote it as $X := \bigsqcup_{U \in \mathcal{P}(V)} L(U)$, and let $i_U : L(U) \hookrightarrow X, U \in \mathcal{P}(V)$, be the canonical injections. Define a relation R on X as follows. Take any $x, x_1 \in X$ of the form $x = i_U(\varphi)$ and $x_1 = i_{U_1}(\varphi_1)$. If $U_1 = U$, then put $(x, x_1) \in R :\Leftrightarrow \varphi_1 = \varphi$; if $U_1 \neq U$, then put

$$(x, x_1) \in R \quad :\Leftrightarrow \sum_{v \in U \Delta U_1} \varphi(v) = \sum_{v \in U \Delta U_1} \varphi_1(v) + p_{\#_4(U \Delta U_1)}, \tag{2}$$

where Δ denotes the symmetric difference between the two subsets.

Observe that if $U \subset V$ is not empty, then $\#L(U) = 2^{N-1}$. Let the dimension d of space \mathcal{H} coincide with this number, $d = 2^{N-1}$. Suppose that A can be viewed as a union of $N + 1$ pairwise disjoint subsets of same cardinality $d : N$ sets denoted as $A_v, v \in V$, and a set \widehat{A} , i.e. $A = (\cup_{v \in V} A_v) \cup \widehat{A}$. Let the elements of A_v be indexed by $L(\{v\})$, and the elements of \widehat{A} by $L(V)$. Write $A_v = \{\Psi_\sigma^v\}_{\sigma \in L(\{v\})}, \widehat{A} = \{F_\pi\}_{\pi \in L(V)}$. Hence, the set A becomes indexed by a subset $X_0 \subset X$, which is a disjoint union of all $L(\{v\}), v \in V$, and $L(V)$. Take any $R' \subset R$ such that $\forall U \in \mathcal{P}(V) \forall \varphi \in L(U) : (i_U(\varphi), i_U(\varphi)) \in R'$. By this one ensures that $\forall U$ and $\forall \varphi, \varphi'$, there is an equivalence $(i_U(\varphi), i_U(\varphi')) \in R' \Leftrightarrow \varphi = \varphi'$. At the same time, unlike the case with R , for $(i_U(\varphi), i_{U_1}(\varphi_1)) \in R', U_1 \neq U$, only an implication ‘ \Rightarrow ’ of the form as above in (2) is valid. The relation R' on X induces a relation on X_0 , which we denote as R'_0 .

Now, suppose that the non-orthogonality relation between the rays that constitute A stems precisely from R'_0 . In other words, $\Psi_\sigma^v \not\perp \Psi_{\sigma_1}^{v_1}$ iff $(i_{\{v\}}(\sigma), i_{\{v_1\}}(\sigma_1)) \in R'_0$, and $\Psi_\sigma^v \not\perp F_\pi$ iff $(i_{\{v\}}(\sigma), i_V(\pi)) \in R'_0$, where $v, v_1 \in V, \sigma \in L(\{v\}), \sigma_1 \in L(\{v_1\}), \pi \in L(V)$. In particular, this implies that the elements of each $A_v, v \in V$, as well as the elements of \widehat{A} are pairwise orthogonal.

The requirement that A is bicolourable yields a condition on the parameters p_0, p_1, p_2, p_3 . Choose and fix a bicolouring, assuming that it exists. In particular, each of the subsets

$A_v, v \in V$, has precisely one red ray, and the subset \widehat{A} has precisely one red ray. Denote the red rays as $\Psi_{\sigma_v}^v, v \in V$, and $F_{\widehat{\pi}}$. Since any two red rays cannot be orthogonal, taking into account the explicit description (2) of R , one obtains

$$\sum_{z=v, v_1} (\sigma_v(z) + \sigma_{v_1}(z)) = p_2, \quad \sum_{z \in V \setminus \{v\}} (\sigma_v(z) + \widehat{\pi}(z)) = p_3.$$

Invoking definitions (1) of $L(U), U \subset V$, one derives

$$\sigma_v(v_1) + \sigma_{v_1}(v) = p_2, \quad \sum_{z \in V \setminus \{v\}} \sigma_v(z) = \widehat{\pi}(v) + p_0 + p_3.$$

Take a sum over all pairs $\{v, v_1\}$ in the first formula. Similarly, take a sum over all v in the second formula. This yields

$$\sum_{\substack{v, v_1 \in V, \\ v_1 \neq v}} \sigma_v(v_1) = \frac{N(N-1)}{2} p_2, \quad \sum_{\substack{v, z \in V, \\ z \neq v}} \sigma_v(z) = \sum_{v \in V} \widehat{\pi}(v) + N(p_0 + p_3).$$

Since $\widehat{\pi} \in L(V)$, we have $\sum_{v \in V} \widehat{\pi}(v) = p_0$. Subtracting the first equality from the second one, we obtain

$$(N+1)p_0 + \frac{N(N-1)}{2} p_2 + Np_3 = 0.$$

Finally, since $N = 4n, n \in \mathbb{N}$, this simply reduces to $p_0 = 0$. Therefore, it is sufficient just to have $p_0 = 1$ in order to claim that the configuration A is non-bicolourable.

It is natural to consider the following problem. Suppose the set A with the described relations between the rays exists. Is it possible to vary its configuration without breaking up these relations? Consider a simple analogy. Take two orthonormal bases $\{e_i\}_{i=0}^3$ and $\{f_j\}_{j=0}^3$ in \mathbb{C}^4 , and suppose that $\mathbb{C}f_j \not\perp \mathbb{C}e_i$ iff $i + j$ is even. The space \mathbb{C}^4 splits into orthogonal sum $H_0 \oplus H_1$, with $H_0 := \text{span}\{e_0, e_2\} = \text{span}\{f_0, f_2\}$ and $H_1 := \text{span}\{e_1, e_3\} = \text{span}\{f_1, f_3\}$. One may rotate infinitesimally the pair $\{f_0, f_2\}$ in such a way that both of its elements are kept in H_0 . At the same time, one may keep the other six vectors fixed, and this will not break up the mentioned description of the non-orthogonality relation. This motivates the following definition. Let A and A' be two configurations of rays in $\mathcal{H}, \dim \mathcal{H} = d$. A bijective map $\delta : A \xrightarrow{\sim} A'$ is called a *deformation* of A , if it satisfies the following condition: for all $l, l' \in A, \delta(l') \not\perp \delta(l)$ iff $l \not\perp l'$. In the following sections we will construct for every $N = 4n, n \in \mathbb{N}$, a non-bicolourable configuration with the non-orthogonality relation stemming from R'_0 as described above, and then prove explicitly that they admit non-trivial deformations. By that a new family of families of non-bicolourable configurations is obtained.

3. Rays and equations

Recall that $N = \#V = 4n, n \in \mathbb{N}$. Assign the values to the parameters: $p_0 = 1, p_1 = p_2 = p_3 = 0$. We need to construct the rays $\Psi_{\sigma_v}^v, v \in V, \sigma \in L(\{v\})$, and $F_{\pi}, \pi \in L(V)$, such that the following implications are valid:

$$\begin{aligned} \Psi_{\sigma}^v \not\perp \Psi_{\sigma_1}^{v_1} &\Rightarrow \sigma(v_1) + \sigma_1(v) = 0, \\ \Psi_{\sigma}^v \not\perp F_{\pi} &\Rightarrow \sum_{z \in V \setminus \{v\}} (\pi(z) + \sigma(z)) = 0. \end{aligned}$$

Put $\mathcal{H} := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ ($N - 1$ times). Let us search the Ψ -rays in the form $\mathbb{C}f_1 \otimes f_2 \otimes \dots \otimes f_{N-1}$, i.e. each of these rays is a one-dimensional subspace spanned

over a *homogeneous* vector. Having in mind the mentioned property of the $\not\perp$ relation with respect to the Ψ -vectors, it is convenient to consider a completely connected non-oriented graph with vertices V . Assign to the edges of this graph the numbers $0, 1, \dots, N - 2$ in such a way that any two edges that share a common vertex are labelled differently. The most natural way of implementing this is the following. Identify V with $\mathbb{Z}/(N - 1) \sqcup \{*\}$, where $*$ is just a formal symbol. Write $\mathbb{Z}/(N - 1)$ additively and identify its elements with $0, 1, \dots, N - 2$. For every $i, j \in \mathbb{Z}/(N - 1), i \neq j$, assign to the edge connecting i and j the number $i + j$. To every edge that connects $*$ and $k \in \mathbb{Z}/(N - 1)$, the number $2k$ is assigned. By that the required numbering of edges is obtained. Now, choose for every edge an orthonormal basis in \mathbb{C}^2 , and let its elements be indexed by $\mathbb{Z}/2$. Denote the basis corresponding to the edge $\{*, k\}, k \in \mathbb{Z}/(N - 1)$, as $\{\varphi[k]_\alpha\}_{\alpha \in \mathbb{Z}/2}$, and the basis corresponding to the edge $\{l, j\}, l, j \in \mathbb{Z}/(N - 1), l \neq j$, as $\{\psi[\{l, j\}]_\beta\}_{\beta \in \mathbb{Z}/2}$. For every $l \in \mathbb{Z}/(N - 1), \sigma \in L(\{l\}), \rho \in L(\{*\})$, put

$$\Psi_\rho^* := \mathbb{C} \bigotimes_{k \in \mathbb{Z}/(N-1)}^{\rho(k)} \varphi[k]_{\rho(k)}, \tag{3}$$

$$\Psi_\sigma^l := \mathbb{C} \left\{ \varphi[l]_{\sigma(*)} \otimes \left(\bigotimes_{\substack{j \in \mathbb{Z}/(N-1), \\ j \neq l}}^{\sigma(j)} \psi[\{l, j\}]_{\sigma(j)} \right) \right\}, \tag{4}$$

where a Feynman–Maslov-type notation is used to determine the place of the factors in a tensor product. For instance, an expression of the form $a^0 \otimes b^2 \otimes c^1$ means nothing more, but $a \otimes c \otimes b$. Note that with this notation the symbols under \otimes -product commute: for example, $a^0 \otimes b^2 \otimes c^1 = b^2 \otimes a^0 \otimes c^1$. It is clear that the required properties of $\not\perp$ relation with respect to Ψ -rays are established.

Now, let us take care about F_π . One can write F_π in the form

$$F_\pi = \mathbb{C} \sum_{\xi \in L(\{*\})} A_\pi(\xi) \bigotimes_{k \in \mathbb{Z}/(N-1)}^{\xi(k)} \varphi[k]_{\xi(k)}, \tag{5}$$

where $A_\pi(\xi) \in \mathbb{C}$ are some coefficients. Note that ξ and π can be completely recovered from their restrictions to $\mathbb{Z}/(N - 1) \subset V$. Denote $|\xi| := \sum_i \xi(i), |\pi| := \sum_i \pi(i), i \in \mathbb{Z}/(N - 1)$. The required implication $\Psi_\xi^* \not\perp F_\pi \Rightarrow |\pi| + |\xi| = 0$ is equivalent to $|\pi| + |\xi| = 1 \Rightarrow \Psi_\xi^* \perp F_\pi$. Hence, $A_\pi(\xi)$ can be non-zero only if $|\pi| + |\xi| = 0$. One has

$$A_\pi(\xi) = 0, \quad \text{if } |\pi| + |\xi| = 1. \tag{6}$$

Therefore, the collection $\{A_\pi(\xi)\}_{\xi, \pi}$ can be viewed as a block-diagonal complex matrix with two non-trivial blocks. One of these blocks corresponds to $|\pi| = |\xi| = 0$, and the other to $|\pi| = |\xi| = 1$. Since the rays $\{F_\pi\}_{\pi \in L(V)}$ are pairwise orthogonal, one has a condition

$$\sum_{\xi: |\xi|=p} A_\pi(\xi)^* A_{\pi'}(\xi) \propto \delta_{\pi, \pi'}, \tag{7}$$

where $|\pi| = |\pi'| = p, p \in \mathbb{Z}/2$. Let us term (7) as the ‘unitarity condition’.

Now consider the implications of the requirements corresponding to the rays $\Psi_\sigma^l, l \in \mathbb{Z}/(N - 1)$ and F_π . If $\Psi_\sigma^l \not\perp F_\pi$, then one must have $(1 + \pi(l)) + \sigma(*) + \sum_{j: j \neq l} \sigma(j) = 0$. Note that since the number of elements in $\mathbb{Z}/(N - 1)$ is odd, the multiplication by 2 is invertible. This allows us to introduce a variable $k = (l + j)/2$ replacing the variable j in the

tensor product in the formula for Ψ_σ^l . One derives: if $\pi(l) + \sigma(*) + \sum_{j:j \neq l} \sigma(j) = 0$, then $F_\pi \perp \Psi_\sigma^l$, i.e.

$$\sum_{\xi \in L(\{*\})} A_\pi(\xi) \delta_{\sigma(*), \xi(l)} \prod_{\substack{k \in \mathbb{Z}/(N-1), \\ k \neq l}} (\psi[\{l, 2k-l\}]_{\sigma(2k-l)}, \varphi[k]_{\xi(k)}) = 0, \tag{8}$$

where (\cdot, \cdot) is the inner product in \mathbb{C}^2 , linear with respect to the second argument. For every $l \in \mathbb{Z}/(N-1)$, this is some collection of equations indexed by π and σ . If one first chooses a value for π , and then for σ , then one has 2^{N-1} possibilities for π and 2^{N-2} for σ . In total there are $(N-1) \times 2^{N-1} \times 2^{N-2}$ equations. It is necessary to define $\varphi[k]_\alpha$ and $\psi[\{l, j\}]_\beta$ in such a way that this system has a solution as a linear system with respect to the indeterminates $A_\pi(\xi)$, $|\pi| = |\xi|$. After that it is necessary to select such a solution that satisfies the unitarity conditions. By that one produces a nonlinear system of equations with respect to the variables $\varphi[k]_\alpha \in \mathbb{C}^2$, $\alpha \in \mathbb{Z}/2$, and $\psi[\{l, j\}]_\beta \in \mathbb{C}^2$, $\beta \in \mathbb{Z}/2$, which played the role of parameters in the system for $A_\pi(\xi)$, and now become indeterminates themselves. It is not obvious that it has any solution at all. Nevertheless, the solutions exist, and in fact there are quite many. They are described in the following sections.

4. Specialization and reduction

The main system of equations (8) in its general form is hardly manageable unless one makes some additional assumptions. Denote

$$u[i, j]_{\alpha, \beta} := (\psi[\{i, j\}]_\beta, \varphi[(i+j)/2]_\alpha), \tag{9}$$

where $i, j \in \mathbb{Z}/(N-1)$, $i \neq j$; $\alpha, \beta \in \mathbb{Z}/2$. The corresponding 2×2 matrices $u[i, j]$ are unitary. Note that $u[i, j] = u[j, i]$. Let us make a crucial assumption, which will let us essentially simplify the equations. Specialize $u[i, j]$:

$$u[i, j] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & x_{i,j} \\ x_{i,j}^* & -1 \end{pmatrix}, \quad |x_{i,j}| = 1. \tag{10}$$

It is important that the parameters $x_{i,j}$ have an absolute value 1. Since $u[i, j] = u[j, i]$, one must mention the condition

$$x_{i,j} = x_{j,i},$$

where i and j vary over $\mathbb{Z}/(N-1)$, $i \neq j$.

Equation (8) for $A_\pi(\xi)$ acquires the form

$$\sum_{\substack{\xi \in L(\{*\}), \\ \xi(l) = \sigma(*)}} A_\pi(\xi) (-1)^{s_l(\sigma, \xi)} \left(\prod_{\substack{k:k \neq l, \\ \sigma(2k-l)=1, \\ \xi(k)=0}} x_{l,2k-l} \right) \left(\prod_{\substack{k:k \neq l, \\ \sigma(2k-l)=0, \\ \xi(k)=1}} x_{l,2k-l}^* \right) = 0,$$

where

$$s_l(\sigma, \xi) := \sum_{k:k \neq l} \delta_{\sigma(2k-l), 1} \delta_{\xi(k), 1};$$

k varies over $\mathbb{Z}/(N-1)$. Multiply the left-hand side of this equation by $\prod_{\substack{k:k \neq l, \\ \sigma(2k-l)=0}} x_{l,2k-l}$.

Invoking the fact that $|x_{i,j}| \equiv 1$, one obtains

$$\sum_{\substack{\xi \in L(\{*\}), \\ \xi(l) = \sigma(*)}} A_\pi(\xi) (-1)^{s_l(\sigma, \xi)} \left(\prod_{\substack{k:k \neq l, \\ \xi(k)=0}} x_{l,2k-l} \right) = 0.$$

Recall that this kind of equation holds for every $l \in \mathbb{Z}/(N - 1)$, every $\pi \in L(V)$ and every $\sigma \in L(\{l\})$, such that $\sigma(*) = \pi(l) + \sum_{j:j \neq l} \sigma(j)$. Since $A_\pi(\xi)$ can be non-zero only if $\xi(l) = |\pi| + \sum_{j:j \neq l} \xi(j)$, one may understand the summation in the latter formula as over all ξ that satisfy $|\xi| = |\pi|$ and $\sum_{j:j \neq l} (\xi(j) + \pi(j) + \sigma(j)) = 0$.

Take any l, π and σ . Observe that if one takes any even number of points of $(\mathbb{Z}/(N - 1)) \setminus \{l\}$, and then changes the values of σ at these points by adding 1, then the result still satisfies $\sigma(*) = \pi(l) + \sum_{j:j \neq l} \sigma(j)$. In particular, consider the points $l + 1$ and $l + 2$, and put $\sigma_1(l + 1) = 1 + \sigma(l + 1)$, $\sigma_1(l + 2) = 1 + \sigma(l + 2)$ and $\sigma_1(v) = \sigma(v)$ for $v \neq l + 1, l + 2$. Take the equation corresponding to (l, π, σ_1) , multiply it by $(-1)^{\sigma(l+1)}$ and add the result to the equation corresponding to (l, π, σ) . By this one obtains

$$\sum_{\substack{\xi \in L(\{*\}), \\ \xi(l) = \sigma(*)}} A_\pi(\xi) \Lambda_l^{(1)}(\sigma, \xi) (-1)^{s_l(\sigma, \xi)} \left(\prod_{\substack{k:k \neq l, \\ \xi(k) = 0}} x_{l, 2k-l} \right) = 0,$$

where

$$\Lambda_l^{(1)}(\sigma, \xi) := 1 + (-1)^{\sigma(l+1)} (-1)^{s_l(\sigma_1, \xi) - s_l(\sigma, \xi)}.$$

Note that for any integer q , one has $(-1)^q = (-1)^{-q}$. Since

$$s_l(\sigma_1, \xi) + s_l(\sigma, \xi) = \sum_{\substack{k:k \neq l, \\ 2k-l \in \{l+1, l+2\}}} (\delta_{\sigma(2k-l), 1} + \delta_{\sigma(2k-l), 0}) \delta_{\xi(k), 1},$$

the expression for $\Lambda_l^{(1)}(\sigma, \xi)$ reduces to

$$\Lambda_l^{(1)}(\sigma, \xi) = 1 + (-1)^{\delta_{\sigma(l+1), 1} + \sum_{s=1, 2} \delta_{\xi(l+s/2), 1}}.$$

Next, transform in a similar way the just obtained system of equations for $A_\pi(\xi)$ (containing $\Lambda_l^{(1)}(\sigma, \xi)$) with respect to the points $(l + 2, l + 3)$, then with respect to $(l + 3, l + 4)$, and so forth until $(l + (N - 3), l + (N - 2))$. This yields

$$\sum_{\substack{\xi \in L(\{*\}), \\ \xi(l) = \sigma(*)}} A_\pi(\xi) \left(\prod_{m=1}^{N-3} \Lambda_l^{(m)}(\sigma, \xi) \right) (-1)^{s_l(\sigma, \xi)} \left(\prod_{\substack{k:k \neq l, \\ \xi(k) = 0}} x_{l, 2k-l} \right) = 0,$$

where

$$\Lambda_l^{(m)}(\sigma, \xi) = 1 + (-1)^{\delta_{\sigma(l+m), 1} + \sum_{s=m, m+1} \delta_{\xi(l+s/2), 1}}.$$

Now let us compute the quantities $\Lambda_l^{(m)}(\sigma, \xi)$. For any $x, y, z \in \mathbb{Z}/2$, the quantity $1 + (-1)^{\delta_{x, 1} + \delta_{y, 1} + \delta_{z, 1}}$ equals 2, whenever $\delta_{x, 1} + \delta_{y, 1} + \delta_{z, 1}$ is even, and 0, otherwise. Hence, $1 + (-1)^{\delta_{x, 1} + \delta_{y, 1} + \delta_{z, 1}} = 2\delta_{x+y+z, 0}$, and one obtains $\Lambda_l^{(m)} = 2\delta_{\sigma(l+m) + \sum_{s=m, m+1} \xi(l+s/2), 0}$. Therefore, the non-trivial contributions to the sum over ξ in the equation for $A_\pi(\xi)$ stem only from those terms, which satisfy

$$\sigma(l + m) + \sum_{s=m, m+1} \xi(l + s/2) = 0, \quad m = 1, 2, \dots, N - 3.$$

Fix σ and l , and put $\xi(l + 1/2) = q, q \in \mathbb{Z}/2$. The rest of the values $\xi(l + (m + 1)/2), m = 2, 3, \dots, N - 2$, become determined. Since q can be only 0 or 1, there are at most two non-trivial terms in the sum.

It is more convenient to index the equations not by σ , but by the non-trivial terms themselves. Take any equation and take any of its non-trivial terms. Denote the corresponding

value of ξ by ξ_0 . The corresponding value σ_0 of σ is then recovered as follows. For $m = 1, 2, \dots, N - 3$, one has $\sigma_0(l + m) = \xi_0(l + m/2) + \xi_0(l + (m + 1)/2)$. The remaining $\sigma_0(l + (N - 2))$ is determined by $\sum_{j:j \neq l} (\xi_0(j) + \pi(j) + \sigma_0(j)) = 0$.

Consider the equation corresponding to (l, π, σ_0) . The second non-trivial term of the sum over ξ corresponds to $\xi = \xi_0^l$, determined by $\xi_0^l(j) = 1 + \xi_0(j)$, $j \in \mathbb{Z}/(N - 1)$, $j \neq l$, and $|\xi_0^l| = |\pi|$. The latter implies $\xi_0^l(l) = \sum_{j:j \neq l} (1 + \xi_0(j)) + |\pi|$. Since N is even, the number of terms in the latter sum over j , which is $N - 2$, is also even. Taking into account that $|\xi_0| = |\pi|$, one obtains $\xi_0^l(l) = \xi_0(l)$. The equation acquires the form

$$X_0^{(l)}(\xi_0)(-1)^{s_l(\sigma_0, \xi_0)} A_\pi(\xi_0) + X_1^{(l)}(\xi_0)(-1)^{s_l(\sigma_0, \xi_0^l)} A_\pi(\xi_0^l) = 0,$$

where $X_p^{(l)}(\xi_0) := \prod_{\substack{k:k \neq l, \\ \xi_0(k)=p}} x_{l,2k-l}$. We need to compute $(-1)^q$, $q := s_l(\sigma_0, \xi_0) + s_l(\sigma_0, \xi_0^l)$,

i.e. to find out when q is even and when it is odd. Substituting the expressions for ξ_0^l , one obtains $q = \sum_{k:k \neq l} \delta_{\sigma_0(2k-l), 1}$. Hence, q is even iff $\sum_{m=1}^{N-2} \sigma_0(l + m) = 0$. But this sum is just $\sum_{j:j \neq l} (\xi_0(j) + \pi(j))$. Invoking the fact $|\pi| = |\xi_0|$, one finally obtains $q = \xi_0(l) + \pi(l)$. The equation, after dropping down the index 0 near ξ_0 and ξ_0^l , becomes

$$\left(\prod_{\substack{k:k \neq l, \\ \xi(k)=0}} x_{l,2k-l} \right) A_\pi(\xi) + (-1)^{\xi(l)+\pi(l)} \left(\prod_{\substack{k:k \neq l, \\ \xi(k)=1}} x_{l,2k-l} \right) A_\pi(\xi^l) = 0. \tag{11}$$

Recall that the parameter l varies over $\mathbb{Z}/(N - 1)$, $\pi \in L(V)$, and ξ is any element of $L(\{*\})$, such that $\sum_{i \in \mathbb{Z}/(N-1)} (\xi(i) + \pi(i)) = 0$; $\xi^l \in L(\{*\})$ is defined by $\xi^l(j) := 1 + \xi(j)$, $j \neq l$, and $\xi^l(l) = \xi(l)$ (of course, $\xi^l(*) = 0$). It is not difficult to solve this system with respect to $A_\pi(\xi)$ and obtain a condition on $\{x_{i,j}\}_{i,j}$ as a condition of solvability. This is implemented in the following sections.

5. Divisibility by 4

Recall that we have made an assumption about the number of points N in $V = (\mathbb{Z}/(N - 1)) \sqcup \{*\}$. It has to be not just even, but divisible by 4, i.e. $N = 4n$, $n \in \mathbb{N}$. Let us find some solutions for the reduced system of equations (11) for $A_\pi(\xi)$ and illustrate how the mentioned assumption emerges.

We have not obtained yet a condition on $\{x_{i,j}\}_{i,j}$ that ensures solvability of the system. Recall that $|x_{i,j}| \equiv 1$. Nevertheless, let us try if it is possible to put all $x_{i,j} = 1$. In this case one has

$$A_\pi(\xi) + (-1)^{\xi(l)+\pi(l)} A_\pi(\xi^l) = 0,$$

for every l and ξ, π , such that $|\xi| = |\pi|$. Rewrite $(-1)^{\xi(l)+\pi(l)}$ as $(-1)^{\delta_{\xi(l), 1+\pi(l)}}$. Note that since $|\xi| = |\pi|$, and since $N - 1$ is odd, the number of points in $\{i \in \mathbb{Z}/(N - 1) | \xi(i) \neq \pi(i)\}$ is even. Hence, for $|\xi| = |\pi|$, one may define

$$a_\pi(\xi) := (-1)^{\frac{1}{2} \sum_i \delta_{\xi(i), 1+\pi(i)}}. \tag{12}$$

Put $a_\pi(\xi) := 0$ in case $|\xi| = 1 + |\pi|$. Claim that $A_\pi(\xi) = a_\pi(\xi)$ is a solution of the considered system. Indeed, one needs to show that

$$(-1)^{q_1/2} + (-1)^{\delta_{\xi(l), 1+\pi(l)}} (-1)^{q_2/2} = 0,$$

where $q_1 := \sum_i \delta_{\xi(i), 1+\pi(i)}$, and $q_2 := \sum_i \delta_{\xi^l(i), 1+\pi(i)}$. But this is equivalent to $1 + (-1)^q = 0$, where

$$q := \delta_{\xi(l), 1+\pi(l)} + \frac{1}{2} \left\{ 2\delta_{\xi(l), 1+\pi(l)} + \sum_{i:i \neq l} [\delta_{\xi(i), 1+\pi(i)} + \delta_{1+\xi(i), 1+\pi(i)}] \right\}.$$

Therefore, one arrives at

$$1 + (-1)^{\frac{N-2}{2}} = 0.$$

The latter is true for $N = 4n$, and false for $N = 4n + 2$.

We have just found a solution of system (11) for $A_\pi(\xi)$ in case all $x_{i,j} = 1$. Now let us prove that the unitarity condition is fulfilled as well. Take any $r \in \mathbb{Z}/2$ and take any π and π' , such that $|\pi| = |\pi'| = r$. Claim

$$\sum_{\xi:|\xi|=r} a_\pi^*(\xi)a_{\pi'}(\xi) = 2^{N-2}\delta_{\pi,\pi'}.$$

Let us first consider the case $\pi = \pi'$. It is necessary to show that $\sum_{\xi:|\xi|=r} (-1)^{\sum_i \delta_{\xi(i),1+\pi(i)}} = 2^{N-2}$. Consider the identity

$$\sum_i (\delta_{\xi(i),1} - \delta_{\pi(i),1}) \equiv \sum_i [\delta_{\xi(i),1}\delta_{\pi(i),0} - \delta_{\xi(i),0}\delta_{\pi(i),1}].$$

One needs to compute the sum

$$\sum_i \delta_{\xi(i),1+\pi(i)} = \sum_i [\delta_{\xi(i),1}\delta_{\pi(i),0} + \delta_{\xi(i),0}\delta_{\pi(i),1}].$$

Taking into account that the signs of the terms of the sum when it stands in the exponent of -1 are unessential, one can apply the above identity and derive

$$(-1)^{\sum_i \delta_{\xi(i),1+\pi(i)}} = (-1)^{\sum_i (\delta_{\xi(i),1} - \delta_{\pi(i),1})} = (-1)^{|\xi|+|\pi|} = 1.$$

Taking the sum over ξ , $|\xi| = r$, one obtains 2^{N-2} , just as required.

Now consider the case $\pi \neq \pi'$. Recall that $|\pi| = |\pi'| = r \in \mathbb{Z}/2$. It is necessary to show that $\sum_{\xi:|\xi|=r} (-1)^{\chi(\xi,\pi,\pi')/2} = 0$, where

$$\chi(\xi, \pi, \pi') := \sum_{k \in \mathbb{Z}/(N-1)} (\delta_{\xi(k),1+\pi(k)} + \delta_{\xi(k),1+\pi'(k)}).$$

Consider a sum $S := \sum_k \delta_{\xi(k),1}[\delta_{\pi(k),1} + \delta_{\pi'(k),1}]$, and rewrite it in two different ways:

$$S = 2 \sum_k \delta_{\xi(k),1} - \sum_k \delta_{\xi(k),1}[\delta_{\pi(k),0} + \delta_{\pi'(k),0}],$$

$$S = \sum_k [\delta_{\pi(k),1} + \delta_{\pi'(k),1}] - \sum_k \delta_{\xi(k),0}[\delta_{\pi(k),1} + \delta_{\pi'(k),1}].$$

Subtracting one equality from the other and regrouping the terms, one obtains an identity

$$\begin{aligned} -2 \sum_k \delta_{\xi(k),1} + \sum_k [\delta_{\pi(k),1} + \delta_{\pi'(k),1}] &= - \sum_k \delta_{\xi(k),1}[\delta_{\pi(k),0} + \delta_{\pi'(k),0}] \\ &+ \sum_k \delta_{\xi(k),0}[\delta_{\pi(k),1} + \delta_{\pi'(k),1}]. \end{aligned}$$

Use this fact to transform $\chi(\xi, \pi, \pi')$:

$$\begin{aligned} \chi(\xi, \pi, \pi') &= \sum_k \sum_{s=0,1} \delta_{\xi(k),s}(\delta_{1+\pi(k),s} + \delta_{1+\pi'(k),s}) \\ &= 2 \sum_k \delta_{\xi(k),1}[\delta_{\pi(k),0} + \delta_{\pi'(k),0}] - 2 \sum_k \delta_{\xi(k),1} + \sum_k [\delta_{\pi(k),1} + \delta_{\pi'(k),1}]. \end{aligned}$$

Taking into account that $(-1)^{\sum_k \delta_{\xi(k),1}} = (-1)^{|\xi|} = (-1)^r$, one reduces the problem of the proof of $\sum_{\xi:|\xi|=r} (-1)^{\chi(\xi,\pi,\pi')/2} = 0$ to

$$\sum_{\xi:|\xi|=r} (-1)^{\sum_k \delta_{\xi(k),1}[\delta_{\pi(k),0} + \delta_{\pi'(k),0}]} = 0.$$

Note that since $\delta_{\pi(k),0} + \delta_{\pi'(k),0}$ is 0 or 2 if $\pi(k) = \pi'(k)$, and 1 otherwise, the sum over k in the latter formula becomes the sum of $\delta_{\xi(k),1}$ over all k , such that $\pi(k) \neq \pi'(k)$. Now look at the sets

$$\{k|\pi(k) = 0\} = \bigsqcup_{p \in \mathbb{Z}/2} \{k|\pi(k) = 0 \ \& \ \pi'(k) = p\},$$

$$\{k|\pi'(k) = 0\} = \bigsqcup_{p \in \mathbb{Z}/2} \{k|\pi(k) = p \ \& \ \pi'(k) = 0\}.$$

The cardinalities of both sets on the left-hand sides viewed in $\mathbb{Z}/2$ are $1 + r$. Apply $\#(\cdot)$ to the left- and right-hand sides and subtract the two equalities. Due to the mentioned fact, the number

$$m := \#\{k|\pi(k) \neq \pi'(k)\}$$

is even. Note that $m \neq N - 1$, since $N - 1$ is odd, and $m \neq 0$, since $\pi \neq \pi'$. For every ξ , $|\xi| = r$, one may consider

$$p := \#\{k|\xi(k) = 1 \ \& \ \pi(k) \neq \pi'(k)\}, \quad q := \#\{k|\xi(k) = 1 \ \& \ \pi(k) = \pi'(k)\}.$$

Since $p + q$ is just the cardinality of $\{k|\xi(k) = 1\}$, one obtains $[p + q]_2 = r$, where $[\cdot]_2 : \mathbb{Z} \rightarrow \mathbb{Z}/2$ is the canonical epimorphism. For any given p and q , $0 \leq p \leq m, 0 \leq q \leq N - 1 - m$, $[p + q]_2 = r$, there exist precisely $C_m^p C_{N-1-m}^q$ ways to choose the corresponding ξ (C_m^p and C_{N-1-m}^q are the binomial coefficients). Hence, the required sum becomes

$$\sum_{\xi:|\xi|=r} (-1)^{\sum_{k:\pi(k) \neq \pi'(k)} \delta_{\xi(k),1}} = \sum_{p=0}^m \sum_{\substack{q=0, \\ [p+q]_2=r}}^{N-1-m} C_m^p C_{N-1-m}^q (-1)^p.$$

All that remains to show is that the expression on the right-hand side vanishes. Denote it by S and rewrite as follows:

$$S = \left(\sum_{\substack{p=0, \\ [p]_2=r}}^m \sum_{\substack{q=0, \\ [q]_2=0}}^{N-1-m} + \sum_{\substack{p=0, \\ [p]_2=1+r}}^m \sum_{\substack{q=0, \\ [q]_2=1}}^{N-1-m} \right) C_m^p C_{N-1-m}^q (-1)^p.$$

Observe that

$$\sum_{\substack{q=0, \\ [q]_2=0}}^{N-1-m} C_{N-1-m}^q = \sum_{\substack{q=0, \\ [q]_2=1}}^{N-1-m} C_{N-1-m}^q,$$

since the difference between these two expressions is equal to $(1 + (-1))^{N-1-m} = 0$ (recall that $m \neq N - 1$). Therefore, S can be written as

$$S = \left(\sum_{\substack{q=0, \\ [q]_2=0}}^{N-1-m} C_{N-1-m}^q \right) \sum_{p=0}^m C_m^p (-1)^p.$$

But the sum over p is just $(1 + (-1))^m = 0$ (recall that $m \neq 0$). Hence $S = 0$, and this completes the proof of the unitarity for $a_\pi(\xi)$.

Note that the fact that 4 divides N has been used only in the proof that $a_\pi(\xi)$ is a solution of the system of equations for $A_\pi(\xi)$. The fact that $\{a_\pi(\xi)\}_{\xi,\pi}$ satisfy the unitarity condition relies just on the assumption that N is even.

6. General case

Note that one can rewrite the system of equations (11) for $A_\pi(\xi)$ as follows:

$$\left(\prod_{\substack{k:k \neq l, \\ \xi_\pi(k)=0}} x_{l,2k-l}^{1-2\delta_{\pi(k),1}} \right) A_\pi(\xi) + (-1)^{\xi_\pi(l)} \left(\prod_{\substack{k:k \neq l, \\ \xi_\pi(k)=1}} x_{l,2k-l}^{1-2\delta_{\pi(k),1}} \right) A_\pi(\xi^l) = 0,$$

where $\xi_\pi(i) := \xi(i) + \pi(i), i \in \mathbb{Z}/(N - 1)$. Therefore it suffices to investigate only the subsystem corresponding to, say, $\pi(i) \equiv 0$. In order to obtain a solution for a general π , one needs to replace each $x_{i,j}$ with $x_{i,j}^{1-2\delta_{\pi((i+j)/2),1}}$, and each $\xi(i)$ with $\xi_\pi(i)$, i.e.

$$A_\pi(\xi) = C_\pi A(\xi_\pi), \quad x_{i,j} \rightarrow x_{i,j}^{1-2\delta_{\pi((i+j)/2),1}}, \tag{13}$$

where $C_\pi \in \mathbb{C}$, and $A(\cdot)$ is $A_\pi(\cdot)$ corresponding to $\pi(i) \equiv 0$. Since the coefficients $A_\pi(\xi)$ in (5) are meant to define the rays F_π , it is necessary to assume $C_\pi \neq 0$. Without loss of generality, $C_\pi = 1$.

Put all $\pi(i) = 0$. Let us derive an expression for $A(\xi)$. Note that for $|\xi| = 1$, equation (6) implies $A(\xi) = 0$. Take any $\xi, |\xi| = 0$. Since $N - 1$ is odd, this implies that $\xi(i)$ takes a value 0 in odd number of points $i \in \mathbb{Z}/(N - 1)$, and 1 in even number of points. Consider the set α of points of $\mathbb{Z}/(N - 1)$ where ξ has the value 0, and equip it with a numbering of the form $\alpha = \{\alpha_0, \alpha_{\pm 1}, \dots, \alpha_{\pm s}\}$ (in total there are $2s + 1$ points). Similarly, consider the set of points of $\mathbb{Z}/(N - 1)$ where the value of ξ is 1, and equip it with a numbering of the form $\beta = \{\beta_{\pm 1}, \dots, \beta_{\pm q}\}$ (in total there are $2q$ points). One has $(2s + 1) + 2q = N - 1$. One may assume that $\alpha_i < \alpha_{i'}$ iff $i < i'$, and $\beta_j < \beta_{j'}$ iff $j < j'$. From the equation for $A(\xi)$, corresponding to $l = \alpha_s$, one obtains

$$A \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} = - \frac{\prod_{k \in \beta} x_{\alpha_s, 2k - \alpha_s}}{\prod_{k \in \alpha \setminus \{\alpha_s\}} x_{\alpha_s, 2k - \alpha_s}} A \begin{pmatrix} 0 \text{ at } \beta \cup \{\alpha_s\}, \\ 1 \text{ at } \alpha \setminus \{\alpha_s\} \end{pmatrix}.$$

Similarly, from the equation for $A(0 \text{ at } \beta \cup \{\alpha_s\}, 1 \text{ at } \alpha \setminus \{\alpha_s\})$, corresponding to $l = \alpha_{-s}$, one obtains

$$A \begin{pmatrix} 0 \text{ at } \beta \cup \{\alpha_s\}, \\ 1 \text{ at } \alpha \setminus \{\alpha_s\} \end{pmatrix} = \frac{\prod_{k \in \alpha \setminus \{\alpha_{\pm s}\}} x_{\alpha_{-s}, 2k - \alpha_{-s}}}{\prod_{k \in \beta \cup \{\alpha_{-s}\}} x_{\alpha_{-s}, 2k - \alpha_{-s}}} A \begin{pmatrix} 0 \text{ at } \alpha \setminus \{\alpha_{\pm s}\}, \\ 1 \text{ at } \beta \cup \{\alpha_{\pm s}\} \end{pmatrix},$$

(this time there is no minus sign before the fraction). It is convenient to denote $y(k, j) := x_{j, 2k-j}$, and put formally $x_{i,i} \equiv 1$. Hence $y(i, i) \equiv 1$. Substituting the latter formula into the formula before it, and using the mentioned formal notation, one obtains

$$A \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} = \left\{ \left(\prod_{k \in \alpha} \frac{y(k, \alpha_{-s})}{y(k, \alpha_s)} \right) \left(\prod_{k \in \beta} \frac{y(k, \alpha_s)}{y(k, \alpha_{-s})} \right) \right\} \frac{-1}{y^2(\alpha_s, \alpha_{-s})} A \begin{pmatrix} 0 \text{ at } \alpha \setminus \{\alpha_{\pm s}\}, \\ 1 \text{ at } \beta \cup \{\alpha_{\pm s}\} \end{pmatrix}.$$

Now, perform the same method, but this time using $\alpha_{\pm(s-1)}$, with the $A(\cdot)$ variable standing on the right-hand side of the latter formula. This yields

$$A \begin{pmatrix} 0 \text{ at } \alpha \setminus \{\alpha_{\pm s}\}, \\ 1 \text{ at } \beta \cup \{\alpha_{\pm s}\} \end{pmatrix} = \left\{ \left(\prod_{k \in \alpha \setminus \{\alpha_{\pm s}\}} \frac{y(k, \alpha_{-(s-1)})}{y(k, \alpha_{s-1})} \right) \left(\prod_{k \in \beta \cup \{\alpha_{\pm s}\}} \frac{y(k, \alpha_{s-1})}{y(k, \alpha_{-(s-1)})} \right) \right\} \\ \times \frac{-1}{y^2(\alpha_{s-1}, \alpha_{-(s-1)})} A \begin{pmatrix} 0 \text{ at } \alpha \setminus \{\alpha_{\pm s}, \alpha_{\pm(s-1)}\}, \\ 1 \text{ at } \beta \cup \{\alpha_{\pm s}, \alpha_{\pm(s-1)}\} \end{pmatrix}.$$

Substitute this result into the above formula and rewrite the expression so that there appear again the products of the form $\prod_{k \in \alpha}$ and $\prod_{k \in \beta}$. One obtains

$$A \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} = \left(\prod_{k \in \{\alpha_{\pm s}\}} \frac{y^2(k, \alpha_{s-1})}{y^2(k, \alpha_{-(s-1)})} \right) \left\{ \prod_{m=s, s-1} \left(\prod_{k \in \alpha} \frac{y(k, \alpha_{-m})}{y(k, \alpha_m)} \right) \left(\prod_{k \in \beta} \frac{y(k, \alpha_m)}{y(k, \alpha_{-m})} \right) \right\} \\ \times \left[\prod_{m=s, s-1} \frac{-1}{y^2(\alpha_m, \alpha_{-m})} \right] A \begin{pmatrix} 0 \text{ at } \alpha \setminus \{\alpha_{\pm s}, \alpha_{\pm(s-1)}\}, \\ 1 \text{ at } \beta \cup \{\alpha_{\pm s}, \alpha_{\pm(s-1)}\} \end{pmatrix}.$$

Proceeding this way, one finally arrives at a formula that has on its right-hand side a variable of the form $A(0_{\alpha_0})$, where 0_{α_0} denotes a function $V \rightarrow \mathbb{Z}/2$ the restriction of which to $\mathbb{Z}/(N-1)$ is 0 only at one point α_0 , and 1 otherwise. The result is of the form

$$A \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} = \left(\prod_{1 \leq m < n \leq s} \frac{y^2(\alpha_n, \alpha_m) y^2(\alpha_{-n}, \alpha_m)}{y^2(\alpha_n, \alpha_{-m}) y^2(\alpha_{-n}, \alpha_{-m})} \right) \\ \times \left\{ \prod_{m=1}^s \left(\prod_{k \in \alpha} \frac{y(k, \alpha_{-m})}{y(k, \alpha_m)} \right) \left(\prod_{k \in \beta} \frac{y(k, \alpha_m)}{y(k, \alpha_{-m})} \right) \right\} \left[\prod_{m=1}^s \frac{-1}{y^2(\alpha_m, \alpha_{-m})} \right] A(0_{\alpha_0}). \tag{14}$$

Recall that $y(k, j) := x_{j, 2k-j}$, $k \neq j$, and $y(i, i) := 1$. It remains to express $A(0_{\alpha_0})$ from the corresponding equation with $l = \alpha_0$. One has

$$A(0_{\alpha_0}) = - \left(\prod_{k=0}^{N-2} y(k, \alpha_0) \right) A(\mathbf{0}), \tag{15}$$

where $\mathbf{0}$ is a function $V \rightarrow \mathbb{Z}/2$, the restriction of which to $\mathbb{Z}/(N-1)$ is a constant function with value 0. Hence, all $A(\xi)$, except $A(\mathbf{0})$, are determined. The value of $A(\mathbf{0})$ remains arbitrary. Since $A(\xi) \neq 0$ is required, without loss of generality, one may put $A(\mathbf{0}) = \mathbf{1}$.

Let us compare these formulae with the result of the previous section. Recall that in order to obtain $A_\pi(\xi)$, $|\xi| = |\pi|$, it is necessary to take an expression for $A(\xi_\pi)$ and replace the parameters $x_{i,j}$ with $x_{i,j}^{1-2\delta_\pi((i+j)/2), 1}$. The latter is equivalent to replacing $y(k, j)$ with $y(k, j)^{1-2\delta_\pi(k), 1}$. If one specializes all $y(k, j)$ to 1, one obtains $A_\pi(\xi) = c_\pi (-1)^{s_\pi+1}$, where $s_\pi := (\#\{i | \xi_\pi(i) = 0\} - 1)/2$, and c_π is an arbitrary constant. One can also define $q_\pi := \#\{i | \xi_\pi(i) = 1\}/2$. Since $2q_\pi + (2s_\pi + 1) = N - 1$, one has $(-1)^{s_\pi+1} = (-1)^{(N-2)/2 - q_\pi + 1}$. Recalling that 4 divides N , one derives that $(N-2)/2$ is odd, and therefore $(-1)^{s_\pi+1} = (-1)^{q_\pi}$. But q_π is precisely $(1/2) \sum_i \delta_{\xi(i), 1+\pi(i)}$, i.e. if one puts $c_\pi = 1$, then $A_\pi(\xi)$ becomes $a_\pi(\xi)$, solution (12) described in the previous section.

Consider any solution of the system for $A_\pi(\xi)$ and separate the factor $a_\pi(\xi)$, i.e. write $A_\pi(\xi) = a_\pi(\xi) \tilde{A}_\pi(\xi)$. The substitution of the latter expression into equation (11) for $A_\pi(\xi)$ corresponding to (l, π, ξ) yields a similar equation for $\tilde{A}_\pi(\xi)$, which looks almost the same and has just one difference: instead of $(-1)^{\xi(l)+\pi(l)}$ one has just the factor -1 in front of the second term. The above considerations imply that any solution $\tilde{A}_\pi(\xi)$ is of the form $\tilde{A}_\pi(\xi) = c_\pi b(\xi)$, where c_π is a constant with respect to ξ , and $b(\xi)$ is some expression of the form $\prod_{i,j} x_{i,j}^{\varepsilon_{i,j}(\xi)}$ with $\varepsilon_{i,j}(\xi) = \pm 1$ being some numbers. Note that the fact $|x_{i,j}| \equiv 1$ implies $|b(\xi)| \equiv 1$. The constants c_π can, of course, be chosen differently for different values of parameters $\{x_{i,j}\}_{i,j}$, but what is important is that $\tilde{A}_\pi(\xi)$ splits into a product of two factors,

one depending only on π , and the other having an absolute value 1 and depending only on ξ . This allows us to establish the unitarity condition in the general case

$$\sum_{\xi:|\xi|=p} A_{\pi}^*(\xi) A_{\pi'}(\xi) = c_{\pi}^* c_{\pi'} \sum_{\xi:|\xi|=p} a_{\pi}^*(\xi) a_{\pi'}(\xi) |b(\xi)|^2 = 2^{N-2} c_{\pi}^* c_{\pi'} \delta_{\pi, \pi'},$$

where $|\pi| = |\pi'| = p$, $p \in \mathbb{Z}/2$.

7. Conditions of existence

It remains to investigate the problem of existence of solutions of the overdetermined linear system of equations for $A_{\pi}(\xi)$. It suffices to consider only the equations for $A(\xi)$, $|\xi| = 0$. For every ξ , let the notation $\alpha = \{\alpha_0, \alpha_{\pm 1}, \dots, \alpha_{\pm s}\}$ and $\beta = \{\beta_{\pm 1}, \dots, \beta_{\pm q}\}$ be as in the previous section. Require that expression (14) for $A(\xi)$ in terms of α and β does not depend on the choice of numbering of the points of α and β . The independence of the numbering of β is seen directly from the formula, so it is necessary to focus on α . The requirement that for any m , $1 \leq m \leq s$, it is possible to interchange the numbers of α_m and α_{-m} yields a condition

$$\begin{aligned} & \left(\prod_{k \in \alpha} \frac{y(k, \alpha_{-m})}{y(k, \alpha_m)} \right) \left(\prod_{k \in \beta} \frac{y(k, \alpha_m)}{y(k, \alpha_{-m})} \right) \frac{1}{y^2(\alpha_m, \alpha_{-m})} \\ &= \left(\prod_{k \in \alpha} \frac{y(k, \alpha_m)}{y(k, \alpha_{-m})} \right) \left(\prod_{k \in \beta} \frac{y(k, \alpha_{-m})}{y(k, \alpha_m)} \right) \frac{1}{y^2(\alpha_{-m}, \alpha_m)}. \end{aligned}$$

Rewrite it as follows:

$$\begin{aligned} & y^2(\alpha_{-m}, \alpha_m) \left(\prod_{k \in \alpha} y^2(k, \alpha_{-m}) \right) \left(\prod_{k \in \beta} y^2(k, \alpha_m) \right) \\ &= y^2(\alpha_m, \alpha_{-m}) \left(\prod_{k \in \alpha} y^2(k, \alpha_m) \right) \left(\prod_{k \in \beta} y^2(k, \alpha_{-m}) \right). \end{aligned}$$

Cancelling out the factor $y^2(\alpha_{-m}, \alpha_m) y^2(\alpha_m, \alpha_{-m})$ and recalling that $y(i, i) \equiv 1$, one obtains $Y_{\alpha, \beta}(\alpha_{-m}, \alpha_m) = Y_{\alpha, \beta}(\alpha_m, \alpha_{-m})$, where

$$Y_{\alpha, \beta}(\alpha_{-m}, \alpha_m) := \left(\prod_{\substack{k \in \alpha, \\ k \neq \alpha_{\pm m}}} y^2(k, \alpha_{-m}) \right) \left(\prod_{k \in \beta} y^2(k, \alpha_m) \right).$$

In particular, since there exists ξ such that the corresponding $\beta = \emptyset$, one has for all $i, j \in \mathbb{Z}/(N-1)$, $i \neq j$,

$$\prod_{k:k \neq i, j} y^2(k, i) = \prod_{k:k \neq i, j} y^2(k, j),$$

where k varies over $\mathbb{Z}/(N-1)$. Now take the case $\alpha = \{\alpha_0, \alpha_{\pm 1}\}$. Then a product $\prod_{k \in \beta}$ is just a product over all k , such that $k \neq \alpha_0, \alpha_{\pm 1}$. If one writes the corresponding expressions and then uses the previous formula with respect to the pair of points $\{\alpha_1, \alpha_{-1}\}$, one obtains $y^2(\alpha_0, \alpha_{-1})/y^2(\alpha_0, \alpha_1) = y^2(\alpha_0, \alpha_1)/y^2(\alpha_0, \alpha_{-1})$. Hence, for any pairwise non-equal i, j and k , one has

$$y^4(k, i) = y^4(k, j).$$

Therefore, each $y(k, j), k \neq j$, is of the form $y(k, j) = \lambda(k)z(k, j)$, where $\lambda(k)$ satisfies $|\lambda(k)| = 1$, and $z(k, j)$ is a fourth root of unity, $z^4(k, j) = 1$. Recall that by definition $y(i, i) \equiv 1$. In particular, if one puts all $z(k, j) = 1$, and substitutes the corresponding $y(k, j)$ into $A(\xi)$, one obtains $A(\xi) = (-1)^{s+1} \tilde{A}(\xi)$:

$$\tilde{A}(\xi) = \left(\prod_{m=1}^s \frac{\lambda(\alpha_m)}{\lambda(\alpha_{-m})} \right) \left(\prod_{m=1}^s \frac{1}{\lambda^2(\alpha_m)} \right) \prod_{k:k \neq \alpha_0} \lambda(k) = \prod_{k \in \beta} \lambda(k).$$

In the general case, taking into account that $(z(k, j))^2 = \pm 1$, and hence $(z(k, j))^2 = 1/(z(k, j))^2$, the expression for $A(\xi)$ can be transformed into

$$A(\xi) = (-1)^{s+1} \left(\prod_{k \in \beta} \lambda(k) \right) \widehat{A}(\xi),$$

where $\widehat{A}(\xi)$ is a monomial in variables $z(k, j)$. Note that the first condition (containing $Y_{\alpha, \beta}(\cdot, \cdot)$) on $y(k, j)$ derived above, after the substitution $y(k, j) = \lambda(k)z(k, j), k \neq j$, yields

$$\prod_{k:k \neq i, j} z^2(k, i) = \prod_{k:k \neq i, j} z^2(k, j). \tag{16}$$

Recall that the second condition $y^4(k, j) = y^4(j, k)$ has reduced to

$$z^4(k, j) = 1, \quad k \neq j, \tag{17}$$

and, by definition, one has $z(i, i) = 1$. Let us derive other conditions.

Consider again the general formulae (14), (15) for $A(\xi)$. Denote by α_{\pm} the sets $\alpha_+ := \{\alpha_1, \alpha_2, \dots, \alpha_s\}, \alpha_- := \{\alpha_{-1}, \alpha_{-2}, \dots, \alpha_{-s}\}$. One may pick any two points with opposite indices, say $\alpha_m \in \alpha_+$ and $\alpha_{-m} \in \alpha_-$, and then interchange their locations, i.e. put α_{-m} in α_+ and α_m in α_- . The conditions we have already derived ensure that $A(\xi)$ remains invariant. Now, for $s \geq 2$, it is necessary to require that it is invariant if one switches the indices of any two points from the same set, i.e. both from α_+ or both from α_- . Let us start with α_1 and α_2 . Look at the general formula for $A(\xi)$. Note that the expression in $\{\cdot \cdot \cdot\}$ brackets, and the expression for $A(0_\alpha)$ are already invariant, so it is necessary to focus on the $(\cdot \cdot \cdot)$ and $[\cdot \cdot \cdot]$ factors. Writing the corresponding equality and cancelling out the common factors, one obtains

$$\frac{y^2(\alpha_2, \alpha_1)y^2(\alpha_{-2}, \alpha_1)}{y^2(\alpha_2, \alpha_{-2})} = \frac{y^2(\alpha_1, \alpha_2)y^2(\alpha_{-2}, \alpha_2)}{y^2(\alpha_1, \alpha_{-2})}.$$

Perform the substitutions $y(k, j) = \lambda(k)z(k, j), k \neq j, k, j = \alpha_1, \alpha_{\pm 2}$, and change the notation for indices $\alpha_1 = p, \alpha_2 = q, \alpha_{-2} = r$. This yields

$$\prod_{\substack{i, j \in \{p, q, r\}, \\ i \neq j}} z^2(i, j) = 1.$$

This equation has to be true for all pairwise distinct p, q, r . Denote $\zeta(i, j) := z^2(i, j)z^2(j, i), i \neq j$. Invoking the agreement $z(k, k) \equiv 1$, put $\forall k : \zeta(k, k) := 1$. Then for any p, q, r , not necessarily pairwise distinct, one has a cocycle-type condition

$$\zeta(p, q)\zeta(q, r)\zeta(r, p) = 1. \tag{18}$$

Note that ζ is symmetric, $\zeta(i, j) = \zeta(j, i)$, and normalized on 1. Hence, $\zeta(i, j)$ in (18) is of the form $\zeta(i, j) = \varphi(i)\varphi(j)$, where $\varphi : \mathbb{Z}/(N - 1) \rightarrow \mathbb{C}$ is some function, such that $\varphi^2(i) \equiv 1$, i.e. $\varphi(i) = \pm 1$. Observe that since $N - 1$ is odd, for any $s \in \mathbb{Z}/(N - 1)$ we have $[s]_2 = 0$ iff $[-s]_2 = 1$ (recall that $[\cdot]_2 : \mathbb{Z}/(N - 1) \rightarrow \mathbb{Z}/2$ denotes the

canonical epimorphism). If one assigns the values of $z^2(k, k + l)$, for all k and, say, only for all $l, [l]_2 = 1$, then one can extend the function $z^2(\cdot, \cdot)$ to all points according to $z^2(k, k - l) = \varphi(k)\varphi(k - l)/z^2(k - l, (k - l) + l)$ (l satisfies $[l]_2 = 1$), $z^2(k, k) = 1$. Since $z^2(i, j)$ can only be ± 1 , and $\varphi(i)$ can only be ± 1 , we have $\varphi(i)z^2(i, j) = \varphi(j)z^2(j, i)$, for all $i, j, i \neq j$. Recalling that $y(k, l) = \lambda(k)z(k, l), k \neq l$, and that there are no conditions on $\lambda(k)$ so for except $|\lambda(k)| = 1$, we see that the factor $\varphi(\cdot)$ is unimportant and can be incorporated into $\lambda(\cdot)$. Therefore, without loss of generality, $y(k, l) = \lambda(k)z(k, l), k \neq l$, and for all i, j

$$z^2(i, j) = z^2(j, i). \tag{19}$$

The latter assumption allows us to rewrite $\widehat{A}(\xi)$ in the form

$$\widehat{A} \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} := \left(\prod_{k \in \alpha} \frac{\prod_{l \in \alpha_-} z(k, l)}{\prod_{l \in \alpha_+} z(k, l)} \right) \left(\prod_{k \in \beta} \frac{\prod_{l \in \alpha_+} z(k, l)}{\prod_{l \in \alpha_-} z(k, l)} \right) \left(\prod_{\substack{i, j \in \alpha \setminus \{\alpha_0\}, \\ i < j}} z^2(i, j) \right) \prod_{k=0}^{N-2} z(k, \alpha_0). \tag{20}$$

It is now clear that we have an expression for $A(\xi)$, which does not feel the choice of numbering of points in α_+ and α_- . Recalling the fact that for any m one may interchange the roles of α_m and α_{-m} , we conclude that $A(\xi)$ does not depend on the choice of partitioning of $\alpha \setminus \{\alpha_0\}$ into α_+ and α_- . It remains to investigate the requirement of no dependence on the choice of the marked point $\alpha_0 \in \alpha$.

Without loss of generality, look at α_0 and α_1 . One obtains

$$\left(\prod_{\substack{j \in \alpha, \\ j \neq \alpha_0, \alpha_1}} z^2(\alpha_1, j) \right) Z_{\alpha, \beta}(\alpha_{-1}, \alpha_1) \prod_k z(k, \alpha_0) = \left(\prod_{\substack{j \in \alpha, \\ j \neq \alpha_0, \alpha_1}} z^2(\alpha_0, j) \right) Z_{\alpha, \beta}(\alpha_{-1}, \alpha_0) \prod_k z(k, \alpha_1),$$

where

$$Z_{\alpha, \beta}(i, j) := \left(\prod_{k \in \alpha} \frac{z(k, i)}{z(k, j)} \right) \left(\prod_{k \in \beta} \frac{z(k, j)}{z(k, i)} \right),$$

$i, j \in \mathbb{Z}/(N - 1)$. Cancelling the common factors and getting rid of denominators, one arrives at

$$z^2(\alpha_1, \alpha_0) \prod_{\substack{i \in \alpha, \\ i \neq \alpha_0, \alpha_1}} z^2(\alpha_1, i) z^2(i, \alpha_0) = z^2(\alpha_0, \alpha_1) \prod_{\substack{i \in \alpha, \\ i \neq \alpha_0, \alpha_1}} z^2(\alpha_0, i) z^2(i, \alpha_1).$$

But the latter equality is implied directly by the assumption $z^2(i, j) \equiv z^2(j, i)$. Therefore, no new conditions on $z(i, j)$ emerge.

We have a well-defined expression for $A(\xi)$, i.e. it does not depend on the choice of numbering of points in $\alpha := \{i \mid \xi(i) = 0\}$ and $\beta := \{j \mid \xi(j) = 1\}$. Still it does not prove that $A(\xi)$ satisfies the whole system equations. It is necessary to verify that the substitution of $A(\xi)$ into the corresponding system indeed turns each of the equations into an identity. This is equivalent to verifying

$$\left(\prod_{\substack{k \in \alpha, \\ k \neq l}} z(k, l) \right) \widehat{A} \begin{pmatrix} 1 \text{ at } \alpha, \\ 0 \text{ at } \beta \end{pmatrix} = \left(\prod_{\substack{k \in \beta, \\ k \neq l}} z(k, l) \right) \widehat{A} \begin{pmatrix} 1 \text{ at } \beta \Delta \{l\}, \\ 0 \text{ at } \alpha \Delta \{l\} \end{pmatrix},$$

for all $l \in \mathbb{Z}/(N - 1)$. With the notation as above, it suffices to consider just two cases: $l = \alpha_0$ and $l = \beta_q$.

Specialize first to the case $l = \alpha_0$. The sets $\alpha' := \beta \Delta \{l\}$ and $\beta' := \alpha \Delta \{l\}$ are of the form $\alpha' = \beta \cup \{\alpha_0\}, \beta' = \alpha \setminus \{\alpha_0\}$. As above, fix some numbering for α and $\beta, \#\alpha = 2s + 1, \#\beta = 2q$. It is natural to equip α' and β' with the following numbering: $\alpha'_0 := \alpha_0, \alpha'_j := \beta_j, j = \pm 1, \pm 2, \dots, \pm q; \beta'_i := \alpha_i, i = \pm 1, \pm 2, \dots, \pm s$. Substitute the corresponding expressions (20) for $\widehat{A}(\cdot)$ into the last equation and simplify the result as follows. First of all, get rid of all denominators and write the equation as an equality between two products of products of the form $\prod_{i,j} z^t(i, j), t = 1$ or 2 . After that, split each of the products $\prod_{i,j} z^t(i, j)$ into several factors of the form $\prod_{i \in I, j \in J} z^t(i, j)$, where I and J are one of the following sets: $\alpha_+, \alpha_-, \beta_+, \beta_-$, or a one-point set $\{\alpha_0\}$. Simplify the expressions using the assumptions $z^2(i, j) = z^2(j, i)$ and $z^4(i, j) = 1$, and invoking the convention $z(i, i) = 1$. Cancel the common factors on the left- and right-hand sides of the equation. Finally, dividing the left over the right-hand side, one arrives at

$$\left(\prod_{\substack{k,j=0 \\ k < j}}^{N-2} \frac{z(k, j)}{z(j, k)} \right) \left[\prod_{\substack{k \in \alpha_- \cup \beta_-, \\ j \in \alpha_+ \cup \beta_+ \cup \{\alpha_0\}}} z^2(k, j) \right] = 1.$$

Note that $z(k, j)/z(j, k)$ is symmetric with respect to the permutation of k and j . Denote the expression in the square brackets by $\Phi(\alpha_- \cup \beta_-)$. Since $z^2(i, j) = z^2(j, i)$ and $z^2(i, i) = z^2(j, j) = 1$, one has $\prod_k z^2(k, i) = \prod_k z^2(k, j)$, where k varies over the entire $\mathbb{Z}/(N - 1)$. Hence, there exists $g \in \{\pm 1\}$, such that for all i one has $\prod_k z^2(k, i) = g$. In particular, for all $k \in \alpha_- \cup \beta_-$ this allows us to derive $z^2(k, \alpha_0) = z^2(\alpha_0, k) = g \prod_{m \in (\alpha_- \cup \beta_-) \cup (\alpha_+ \cup \beta_+)} z^2(m, k)$. Expressing $z^2(k, \alpha_0)$ this way in $\Phi(\alpha_- \cup \beta_-)$, regrouping the factors and taking into account that $\#(\alpha_- \cup \beta_-) = (N - 2)/2$ is odd, one derives

$$\Phi(\alpha_- \cup \beta_-) = g \left(\prod_{\substack{k \in \alpha_- \cup \beta_-, \\ j \in \alpha_+ \cup \beta_+}} z^4(k, j) \right) \prod_{\substack{k, j \in \alpha_- \cup \beta_-, \\ k < j}} z^4(k, j) = g.$$

Therefore, one has a condition $\prod_{k, j: k < j} z(k, j)/z(j, k) = g$. Squaring the left- and the right-hand sides, taking a product over all k and using the assumptions about $z^2(i, j)$, one derives $\prod_{k, j=0}^{N-2} z^2(k, j) = g^2 = 1$. On the other hand, the latter product is just $\prod_{k=0}^{N-2} (\prod_{j=0}^{N-2} z^2(k, j)) = g^{N-1} = g$. Therefore, $g = 1$. Recall that $z(k, j)/z(j, k)$ is symmetric with respect to the permutation of j and k . Hence, one may take any relation Q on $\mathbb{Z}/(N - 1)$, such that $\forall k, j, j \neq k : (k, j) \in Q \Leftrightarrow (j, k) \notin Q$, and rewrite the product in the (\dots) factor in the above condition as $\prod_{(k, j) \in Q} z(k, j)/z(j, k)$. In particular, it is possible to take Q being formed by all ordered pairs (k, j) , such that $[j - k]_2 = 1$ (recall that since $N - 1$ is odd, $[j - k]_2 = 1$ iff $[k - j]_2 = 0$). Then the condition on $z(k, j)$ acquires the form

$$\prod_{\substack{k, j \in \mathbb{Z}/(N-1), \\ [j-k]_2=1}} z(k, j) = \prod_{\substack{k, j \in \mathbb{Z}/(N-1), \\ [j-k]_2=1}} z(j, k). \tag{21}$$

But this condition implies that $\prod_{j: j \neq k} z^2(k, j)$ equals

$$\prod_{m: [m]_2=1} z^2(k, k + m) z^2(k, k - m) = \prod_{m: [m]_2=1} z^4(k, k + m) = 1,$$

i.e. condition (16) is implied by (17), (19), (21).

Now consider the case $l = \beta_q$. Essentially it is investigated as the case above. For this l one has $\alpha'' := \beta \Delta \{l\} = \beta \setminus \{\beta_q\}$, $\beta'' := \alpha \Delta \{l\} = \alpha \cup \{\beta_q\}$. Hence $\#\alpha'' = 2q - 1$, and $\#\beta'' = 2s + 2$. It is natural to put $\alpha''_0 = \beta_{-q}$, $\alpha''_+ = \beta_+ \setminus \{\beta_q\}$, $\alpha''_- = \beta_- \setminus \{\beta_{-q}\}$, $\beta''_+ = \alpha_+ \cup \{\alpha_0\}$ and $\beta''_- = \alpha_- \cup \{\beta_q\}$. The following equality is required:

$$\left(\prod_{\substack{k \in \alpha, \\ k \neq \beta_q}} z(k, \beta_q) \right) \widehat{A} \begin{pmatrix} 0 \text{ at } \alpha, \\ 1 \text{ at } \beta \end{pmatrix} = \left(\prod_{\substack{k \in \beta, \\ k \neq \beta_q}} z(k, \beta_q) \right) \widehat{A} \begin{pmatrix} 0 \text{ at } \alpha'', \\ 1 \text{ at } \beta'' \end{pmatrix}.$$

Substitute the corresponding expressions for $\widehat{A}(\cdot)$ (see (20)) into this equality and then transform it as follows. Get rid of the denominators, and then rearrange the factors on the left- and right-hand sides so that each becomes a product of products of the form $\prod_{i \in I, j \in J} z^t(i, j)$, where $t = 1$ or 2 , and I and J are one of the following sets: α_+ , α_- , $\beta_+ \setminus \{\beta_q\}$, $\beta_- \setminus \{\beta_{-q}\}$, $\{\alpha_0\}$, $\{\beta_q\}$ or $\{\beta_{-q}\}$. Simplify the expressions using $z(i, i) = 1$, $z^2(i, j) = z^2(j, i)$ and $z^4(i, j) = 1$. Finally, divide the left-hand side over the right-hand side, and simplify the result again. A brute force calculation yields

$$\left(\prod_{\substack{k, j=0, \\ k < j}}^{N-2} \frac{z(k, j)}{z(j, k)} \right) \Phi((\beta_- \setminus \{\beta_{-q}\}) \cup (\alpha_- \cup \{\beta_q\})) = 1.$$

But the Φ factor on the left-hand side is itself equal to 1, so one arrives just at condition (21) already derived above.

Finally, it remains to recall that $x_{i,j} = \lambda((i+j)/2)z((i+j)/2, j)$, $i \neq j$, and that $\forall i, j, i \neq j : x_{i,j} = x_{j,i}$. This yields $z((i+j)/2, i) = z((i+j)/2, j)$, or, what is equivalent, $\forall k, m : z(k, k+m) = z(k, k-m)$. Rewriting condition (21) on z in the form $\prod_k \prod_{m:|m|_2=1} z(k, k+m) = \prod_k \prod_{m:|m|_2=1} z(k, k-m)$, one can see that it is satisfied. Next, since the square $z^2(i, j)$ is symmetric with respect to the permutation of i and j , one can always represent $z(k, j)$ in the form $z(k, j) = \varkappa(2k-j, j)\mu(k, j)$, where $\forall k, j : \mu(k, j) = \mu(j, k)$, and $\forall i, j : \varkappa^2(i, j) = 1$. One has $\mu^4(k, j) = 1$ and $\mu^2(k, k+m) = \mu^2(k, k-m)$. It is always possible to adjust $\mu(\cdot, \cdot)$ and $\varkappa(\cdot, \cdot)$ so that $\forall k, m : \mu(k, k+m) = \mu(k, k-m)$. The formula for $x_{i,j}$, $i \neq j$, now becomes

$$x_{i,j} = \varkappa(i, j)\lambda\left(\frac{i+j}{2}\right)\mu\left(\frac{i+j}{2}, j\right). \tag{22}$$

Since $x_{i,j} = x_{j,i}$, there is a condition $\forall i, j : \varkappa(i, j) = \varkappa(j, i)$. Note that if one takes any μ and \varkappa with the mentioned properties, and puts $z(k, j) = \varkappa(2k-j, j)\mu(k, j)$, then one has automatically $\forall k, m : z(k, k+m) = z(k, k-m)$, together with all other required properties.

The functions μ and \varkappa in (22) can be interpreted as follows. Consider a unit circle in \mathbb{C} centred at zero, and mark on it the points $e^{i2\pi m/(N-1)}$, $m = 0, 1, \dots, N-2$. Identify naturally these points with the elements of $\mathbb{Z}/(N-1)$. Look at all chords connecting the marked points. To define a function μ is the same as to define a function on all these chords with values in the roots of unity of degree 4. Whenever two chords have the same length and share a common vertex, the corresponding values of the function must coincide. Note that if $N-1$ is prime, then this implies that the value $\mu(i, j)$ is determined by $j-i$. The definition of \varkappa in (22) is equivalent to assigning the numbers ± 1 to the chords in an arbitrary manner.

This completes the investigation of the conditions of solvability of the system for $A_\pi(\xi)$. Since these conditions can be satisfied, one is able to construct new examples of non-bicolourable configurations of rays. Let us summarize the results. Recall that the described

construction involves a set $V := (\mathbb{Z}/(N-1)) \sqcup \{*\}$, where $*$ is a formal symbol. To every $v \in V$, a collection of rays $\Psi_\sigma^v, \sigma \in L(\{v\})$, in $(\mathbb{C}^2)^{\otimes(N-1)}$ is assigned, and there is also one more collection of rays $F_\pi, \pi \in L(V)$, defined in terms of $A_\pi(\xi)$. Invoking the notation (1) and that $p_0 = 1, p_1 = 0$, one has $L(\{v\}) = \{\phi : V \rightarrow \mathbb{Z}/2 \mid \phi(v) = 0\}$, and $L(V) = \{\phi : V \rightarrow \mathbb{Z}/2 \mid \sum_{z \in V} \phi(z) = 1\}$.

Theorem 1. *Let $N \in \mathbb{N}$ be a positive integer divisible by 4. Let $\lambda : \mathbb{Z}/(N-1) \rightarrow \mathbb{C}$ and $\mu, \varkappa : (\mathbb{Z}/(N-1))^2 \rightarrow \mathbb{C}$ be any functions having the values on a unit circle, in roots of unity of degree 4, and in roots of unity of degree 2, respectively. Assume that both μ and \varkappa are symmetric with respect to the permutation of their two arguments, and that $\forall j, k : \mu(k, k+j) = \mu(k, k-j)$. Let $\{\varphi[k]_\alpha\}_{\alpha \in \mathbb{Z}/2, k \in \mathbb{Z}/(N-1)}$, be any family of orthonormal bases in \mathbb{C}^2 . For every $i \neq j$, put*

$$x_{i,j} := \varkappa(i, j) \lambda\left(\frac{i+j}{2}\right) \mu\left(\frac{i+j}{2}, j\right).$$

Define the rays $\Psi_\rho^v, \rho \in L(\{v\}), v \in V := (\mathbb{Z}/(N-1)) \sqcup \{*\}$, according to (3), (4), using (9), (10). Define the rays $F_\pi, \pi \in L(V)$, according to (5), using (6), (13), (14), (15). Claim that the finite configuration in $(\mathbb{C}^2)^{\otimes(N-1)}$ formed by these rays is non-bicolourable.

8. Discussion

In the present paper a new infinite family of examples of non-bicolourable configurations of rays has been described. More precisely, it is better to view it as a family of families of examples: there is a parameter $N = 4n, n \in \mathbb{N}$, indexing the families, and for each N there are several complex-valued parameters (their number depends on N), which index the configurations. If one puts $N = 4$ and all $x_{i,j} = 1$, then one recovers just the configuration of rays described in [15, 16], but in a completely different notation.

Observe that the projective lines $\Psi_\rho^*, \rho \in L(\{*\})$, do not depend on $x_{i,j}$. The relation $\not\perp$ depends on $x_{i,j}$, but at the same time it does not change if one varies the continuous part of $x_{i,j}$, i.e. the function λ . Hence, for a given N , the corresponding configurations we have can be viewed as deformations of each other in the sense of the definition given above. It is natural to mark the configuration corresponding to all $\lambda(k) = 1$, and view the others as its deformations.

We have a notion of deformation which connects two configurations. Its definition implies that either both configurations are bicolourable, or both are non-bicolourable. It is possible to extend this definition so that to capture the transition *bicolourable* \rightarrow *non-bicolourable*, and vice versa, but this stays beyond the framework of the present paper. It is, of course, necessary to replace the requirement that the bijection in the definition of deformation respects the $\not\perp$ relation by something else. Intuitively, this should be a requirement that the bijection respects the ‘template’ of the relation, but not the relation itself. For example, in the definition of the relation R there are four parameters p_0, p_1, p_2, p_3 . Varying the values of the parameters, one obtains different R , but of a similar form. In order to formulate a consistent generalization, it appears natural to consider *saturated* configurations, i.e. such ones, for which any subset of pairwise orthogonal rays can be embedded in a subset of d pairwise orthogonal rays, d —the dimension of space \mathcal{H} . So it is necessary to construct a saturation for the new examples first.

The other direction of possible generalizations is to increase the number of colours. Note that it is related to the interpretation of the Kochen–Specker theorem in terms of generalized valuations in [20–23]. A reasonable definition of *non-colourable* configuration for several colours will, apparently, require an introduction of some filtration on $\mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H})$, which is meant to replace the $\not\perp$ relation. This filtration might depend on a particular physical problem.

The set of colours, present in the definition, would also require some additional structure, similar to that of an orthoalgebra.

Intuitively, a saturated configuration has to possess some symmetry. For the configuration corresponding to $N = 4$ and all $x_{i,j} = 1$, a finite saturation has been constructed in [24], motivated by [25]. Its symmetry turns out to be described by a non-trivial Abelian extension \mathcal{G} of $(\mathbb{Z}/2)^6$ over $GL(4, \mathbb{F}_2)$ (the general linear group of 4×4 matrices over a field with two elements). The number 6 stems from some combinatorics and should be viewed as a binomial coefficient C_4^2 . Hence, one looks at a short exact sequence in the category of groups

$$0 \rightarrow (\mathbb{Z}/2)^6 \rightarrow \mathcal{G} \rightarrow GL(4, \mathbb{F}_2) \rightarrow 0. \quad (23)$$

The point of view described in that paper appears to be quite general. It might be possible to construct new finite saturated non-bicolourable configurations by way of considering in analogy with (23) the extensions $0 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 0$, for G and A being some other finite groups, and A , say, elementary Abelian. The condition of non-bicolourability then yields simply a condition on the corresponding cohomology class $\alpha \in H^2(G, A)$.

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